P306.

6. Sol:

Set $y = x^{-\frac{1}{2}} v$

then $y' = -\frac{1}{2} x^{-\frac{3}{2}} v + x^{-\frac{1}{2}} v'$.

$y'' = \frac{3}{4} x^{-\frac{5}{2}} v - x^{-\frac{3}{2}} v' + x^{-\frac{1}{2}} v''$.

$\Rightarrow x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$

$= \cancel{\frac{3}{4} x^{-\frac{1}{2}} v} - \cancel{x^{\frac{1}{2}} v'} + x^{\frac{3}{2}} v'' \cancel{- \frac{1}{2} x^{-\frac{1}{2}} v} + \cancel{x^{\frac{1}{2}} v'} + x^{\frac{3}{2}} v \cancel{- \frac{1}{4} x^{-\frac{1}{2}} v}$

$= x^{\frac{3}{2}} (v'' + v)$

it's clear that $v_1(x) = \cos x$, $v_2(x) = \sin x$ are sol's

i.e. $y_1 = x^{-\frac{1}{2}} \cos x$, $y_2 = x^{-\frac{1}{2}} \sin x$ are sol's.

8. Sol:

a) Obs: $\lim_{x \to 0} x \cdot \frac{x}{x^2} = 1$

$\lim_{x \to 0} x^2 \cdot \frac{x^2 - \nu^2}{x^2} = -\nu^2$.

i.e. $x=0$ is a regular singular pt.

the indicial equ is: $\rho(\rho - 1) + \rho - \nu^2 = 0$.

i.e. $\rho_1 = \nu$, $\rho_2 = -\nu$ are the roots.

b) by the standard thm, the equ has sol in form $y = \sum_{k=0}^{\infty} C_k x^{k+\rho}$ $(C_0 \neq 0)$

substitute into the equ, we get

$$\sum_{k=0}^{\infty} \left[(\rho + \nu + k)(\rho - \nu + k) C_k + C_{k-2} \right] x^{k+\rho} = 0.$$

where we set $C_{-1} = C_{-2} = 0$.

i.e. we get

$$(\rho+\nu+k)(\rho-\nu+k)\,C_k + C_{k-2} = 0 . \quad (k=0,1,2,\cdots).$$

Since we have $C_0 \neq 0$, & $C_{-1} = C_{-2} = 0$, set $k=0$,

we get $(\rho+\nu)(\rho-\nu) = 0$.

i.e. $\rho_1 = \nu$, $\rho_2 = -\nu$ are two indicial roots.

for $\rho = \rho_1 = \nu$, the recursion formula becomes

$$(2\nu+k)\,k\,C_k + C_{k-2} = 0 . \quad (k=1,2,\cdots)$$

take $k=1$, then $(2\nu+1)\,C_1 = 0$. $\Rightarrow C_1 = 0$

$k=3$, $3(2\nu+3)\,C_3 + C_1 = 0 \quad \Rightarrow C_3 = 0$

$\cdots$

i.e. $C_{2l+1} = 0$, $\forall\, l \in \mathbb{N}_0$.

Now, take $k=2$, we get

$$2(2\nu+2)\,C_2 + C_0 = 0 \quad \Rightarrow C_2 = \frac{-1}{2^2(1+\nu)}\,C_0$$

$$k=4 \quad \Rightarrow C_4 = \frac{1}{2^4(\nu+1)(\nu+2)\,2!}\,C_0$$

$\cdots$

$$C_{2l} = \frac{(-1)^l}{2^{2l}(\nu+1)(\nu+2)\cdots(\nu+l)\,l!}\,C_0$$

Recall the properties of Γ function ($\Gamma(1)=1$, $\Gamma(1+x) = x\,\Gamma(x)$)

$$\Rightarrow \Gamma(\nu+l+1) = (\nu+l)(\nu+l-1)\cdots(\nu+2)(\nu+1)\,\Gamma(\nu+1).$$

$$\Gamma(1+l) = l!$$

take $C_0 := \dfrac{1}{2^\nu\,\Gamma(1+\nu)}$, then $C_{2l} = \dfrac{(-1)^l}{2^{2l+\nu}\,\Gamma(\nu+l+1)\,\Gamma(1+l)}$

hence, we get a sol of Bessel's equ corresponding to $\rho=\nu$:

$$y = J_\nu(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(\nu+l+1)\,\Gamma(l+1)} \left(\frac{x}{2}\right)^{2l+\nu}$$

c> For the indicial root $\rho = -\nu$, the recursion formula becomes

$$k(k-2\nu)\, C_k + C_{k-2} = 0, \qquad k = 1, 2, \cdots.$$

if $2\nu \notin \mathbb{Z}$, then $k - 2\nu \neq 0$,

by the arguments before, we obtain

$$C_{2l+1} = 0, \qquad \forall\, l \in \mathbb{N}_0.$$

$k=2 \Rightarrow \quad 2(2-2\nu)\, C_2 + C_0 = 0$

$\Rightarrow \quad C_2 = \dfrac{-1}{2^2(-\nu+1)} C_0$

$k=4 \Rightarrow \quad C_4 = \dfrac{(-1)^2}{2^4(2-\nu)(1-\nu)\,2!} C_0$

$\cdots$

$$C_{2l} = \frac{(-1)^l}{2^{2l}(-\nu+1)(-\nu+2)\cdots(-\nu+l)\,l!} C_0$$

$$= \frac{(-1)^l\, \Gamma(1-\nu)}{2^{2l}\,\Gamma(-\nu+l+1)\,\Gamma(1+l)} C_0.$$

take $C_0 = \dfrac{1}{2^{-\nu}\,\Gamma(1-\nu)}$

we get another sol: $y = J_{-\nu}(x) = \displaystyle\sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(-\nu+l+1)\,\Gamma(l+1)} \left(\frac{x}{2}\right)^{2l-\nu}.$

P315.

16. Sol:

$$\mathcal{L}[f] = \int_0^\infty e^{-st}\, t \sin at\, dt = -\frac{1}{s}\int_0^\infty t \sin at\, d(e^{-st})$$

$$= \frac{1}{s} e^{-st}\, t\sin at\Big|_\infty^0 + \frac{1}{s}\int_0^\infty e^{-st}\, d(t \sin at) \qquad (\text{need } s>0)$$

$$= \frac{1}{s}\int_0^\infty e^{-st}(\sin at + at\cos at)\, dt = -\frac{1}{s^2}\int_0^\infty \sin at + at\cos at\, de^{-st}$$

$$= \frac{1}{s^2}\int_0^\infty e^{-st}\, d(\sin at + at\cos at) = \frac{1}{s^2}\int_0^\infty e^{-st}(2a\cos at - a^2 t\sin at)\, d$$

Notice that

$$\int_0^\infty e^{-st}\cos at\,dt$$

$$= -\frac{1}{s}\int_0^\infty \cos at\,de^{-st} = \frac{1}{s}e^{-st}\cos at\Big|_\infty^0 + \frac{1}{s}\int_0^\infty e^{-st}\,d(\cos at)$$

$$= \frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt$$

$$= \frac{1}{s} + \frac{a}{s^2}\int_0^\infty \sin at\,de^{-st} = \frac{1}{s} - \frac{a}{s^2}\int_0^\infty e^{-st}\,d(\sin at)$$

$$= \frac{1}{s} - \frac{a^2}{s^2}\int_0^\infty e^{-st}\cos at\,dt$$

$$\Rightarrow \int_0^\infty e^{-st}\cos at\,dt = \frac{1}{s}\cdot\frac{s^2}{s^2+a^2} = \frac{s}{s^2+a^2}$$

hence $\left(1+\frac{a^2}{s^2}\right)\int_0^\infty e^{-st}\,t\sin at\,dt$

$$= \frac{2a}{s^2}\int_0^\infty e^{-st}\cos at\,dt = \frac{2a}{s(s^2+a^2)}$$

i.e. $\int_0^\infty e^{-st}\cos at\,dt = \frac{2as}{(s^2+a^2)^2}, \quad s>0.$

18. Sol:

$$\int_0^\infty e^{-st}t^n e^{at}\,dt = \int_0^\infty e^{(a-s)t}t^n\,dt$$

$$= \frac{1}{a-s}\int_0^\infty t^n\,d e^{(a-s)t} = \frac{1}{a-s}e^{(a-s)t}t^n\Big|_0^\infty - \frac{1}{a-s}\int_0^\infty n t^{n-1}e^{(a-s)t}\,dt$$

(we need $a-s<0$ i.e. $s>a$.)

$$= -\frac{n}{a-s}\int_0^\infty t^{n-1}e^{(a-s)t}\,dt = \frac{n(n-1)}{(a-s)^2}\int_0^\infty t^{n-2}e^{(a-s)t}\,dt$$

$$= \cdots = \frac{(-1)^n n!}{(a-s)^n}\int_0^\infty e^{(a-s)t}\,dt = \frac{(-1)^{n+1}n!}{(a-s)^{n+1}} = \frac{n!}{(s-a)^{n+1}}.\ (s>a)$$

20. Sol:

$$\int_0^\infty e^{-st}\, t^2\, \frac{e^{at}-e^{-at}}{2}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\, t^2 e^{at}\, dt - \frac{1}{2}\int_0^\infty e^{-st}\, t^2 e^{-at}\, dt$$

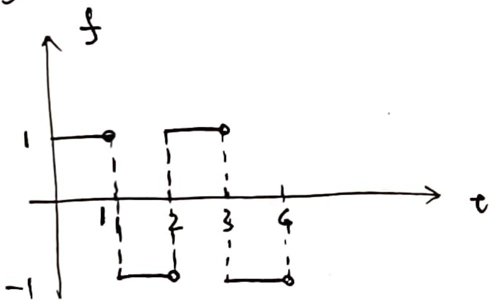
$$= \frac{1}{2}\cdot\frac{2!}{(s-a)^{2+1}} - \frac{1}{2}\cdot\frac{2!}{(s+a)^{2+1}}$$

$$= (s-a)^{-3} - (s+a)^{-3}. \qquad (\text{need } s > |a|)$$

$$= \frac{2a(3s^2+a^2)}{(s^2-a^2)^3} \qquad (s > |a|).$$

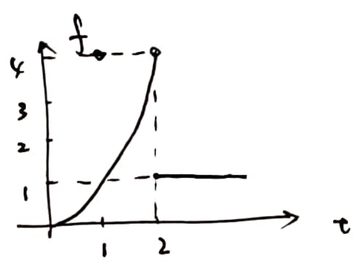
P_{333}.

8. Sol:



$$f(t) = 1 - 2u_1(t) + 2u_2(t) - 2u_3(t) + u_4(t).$$

10. Sol:



$$f(t) = t^2 + (1-t^2)\, u_2(t)$$

34. Sol:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} e^{-st} f(t)\, dt$$

$$= \sum_{k=0}^{\infty} \int_0^T e^{-s(x+kT)} f(x+kT)\, dx$$

$$= \sum_{k=0}^{\infty} e^{-skT} \int_0^T e^{-sx} f(x)\, dx$$

$$= \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$$